Systems of linear equations

All of linear algebra can be thought of as the study of linear equations. What is a linear equation?

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where $a_1, \dots, a_n, b$ are coefficients (some real numbers) and $x_1, \dots, x_n$ are variables.

Ex: 1.) $x - 2y + z = 4$

2.) $2(\sqrt{3} + x) - y = z$ can be rearranged as

$$2x - y - z = -2\sqrt{3}$$

3.) $x(4 - y) = 3$ is not linear

$$4x - \underbrace{xy}_{\substack{\text{quadratic term} \\ \text{(not linear)}}} = 3$$

Linear equations are ubiquitous — model any quantities that have a linear relationship.

Ex: A collection of quarters and nickels has a value of \$1.25. How many of each?

Set q = # of quarters, n = # of nickels.

$$0.25q + 0.05n = 1.25.$$

Can't solve yet – need more info. Say there are 9 coins total.

$$q + n = 9 \Rightarrow q = 9 - n.$$

$$0.25(9-n) + 0.05n = 1.25 \overset{\text{solve}}{\Longrightarrow} n = 5,\ q = 4.$$

This is a simple, boring example, but systems of linear equations (i.e. multiple linear equations) model situations in nearly every area of science, social science, and of course math. Linear algebra is the tool needed to work with and understand these much more complicated systems of equations.

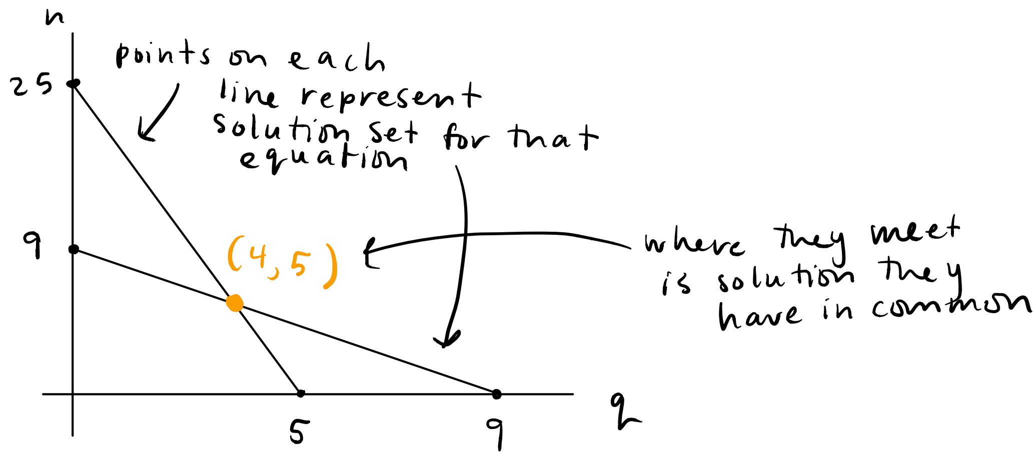
Def: 1.) A <u>system of linear equations</u> is a collection of one or more linear equations involving the same variables (say $x_1, \dots, x_n$).

2.) A <u>solution</u> of the system is a list $(s_1, \dots, s_n)$ of numbers that make each equation true when substituted for $x_1, \dots, x_n$.

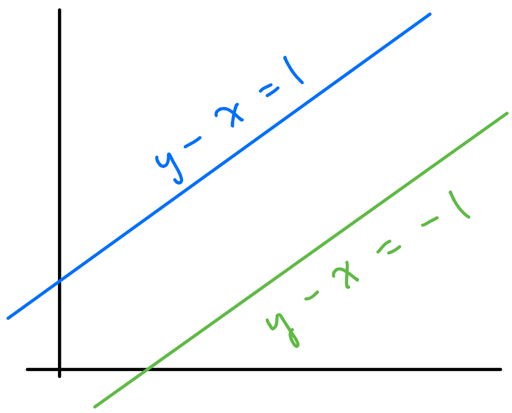
Ex: The system above is
$$\begin{aligned} 0.25q + 0.05n &= 1.25 \\ q + n &= 9 \end{aligned}$$

The (only) solution is $q = 4$, $n = 5$ or $(4, 5)$.

We can also see this geometrically by graphing the two equations:



If the two lines don't meet, that means there are no solutions, i.e. when the lines are parallel:



If the lines coincide, there are infinitely many solutions:

$x_1 + x_2 = 4$, $\quad 3x_1 + 3x_2 = 12$

same solution set

In general, we will see that there are exactly 3 possibilities.

A system of eqns has either:

- no solution $\}$ system is <u>inconsistent</u>
- exactly one solution, or
- infinitely many solutions.

$\}$ system is <u>consistent</u>

If there are more than two variables, it becomes more complicated to describe the solutions geometrically. Instead, we write them algebraically, in parametric form.

Ex: Consider the system

$$\begin{aligned} x + y + z &= 3 \\ 2y - z &= 5. \end{aligned}$$

Adding the equations, we get

$$x + 3y = 8 \Rightarrow x = 8 - 3y.$$

We can also solve for z in terms of y: $z = 2y - 5$.

Then if $y = t$ (some arbitrary value), we get solutions

$$\begin{aligned} x &= 8 - 3t \\ y &= t \\ z &= 2t - 5 \end{aligned}$$

This solution is in <u>parametric form</u>. t is a <u>parameter</u> (sometimes we will need multiple parameters).

Augmented matrices

Augmented matrices simplify both the notation and computations involved in solving systems of equations.

Ex: Consider
$$\begin{aligned} 3x_1 + 2x_2 - x_3 + x_4 &= -1 \\ 2x_1 \qquad - x_3 + 2x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 + 5x_4 &= 2. \end{aligned}$$

We can record the data of this system (the coefficients) in an <u>augmented matrix</u>:

$$\underbrace{\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}}_{\text{coefficient matrix}} \left| \underbrace{\begin{matrix} -1 \\ 0 \\ 2 \end{matrix}}_{\text{constant matrix}} \right]$$

Two systems are <u>equivalent</u> if they have the same solution set. To solve a system, we modify our original system to get an equivalent system until it is easy to solve.

Ex:
$$\begin{aligned} x + 3y &= 7 \\ 2x + y &= -1 \end{aligned} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{②-2①} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -15 \end{array}\right] \xrightarrow{-\frac{1}{5}②} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{①-3②} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right] \rightsquigarrow \begin{aligned} x &= -2 \\ y &= 3 \end{aligned}$$

In each step above, we performed an <u>elementary operation</u>, which results in an equivalent system.

The following are the three elementary row operations that you can perform on an augmented matrix when solving a linear sys. of eqns.:

I. Interchange two rows.

II. Multiply one row by a nonzero number.

III. Add a multiple of one row to a different row.

Eventually, we hope to get a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$$

or, in 3 variables $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$.

Ex:

$$\begin{aligned} 2x - 4y + 5z &= 10 \\ y + 2z &= -1 \\ x - 3y + z &= 2 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & -4 & 5 & 10 \\ 0 & 1 & 2 & -1 \\ 1 & -3 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{1 - 2\,3} \left[\begin{array}{ccc|c} 0 & 2 & 3 & 6 \\ 0 & 1 & 2 & -1 \\ 1 & -3 & 1 & 2 \end{array}\right] \xrightarrow[\text{1, 3}]{\text{swap}} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{3 - 2(2)} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 8 \end{array}\right] \xrightarrow{1 + 3(2)} \left[\begin{array}{ccc|c} 1 & 0 & 7 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 8 \end{array}\right]$$

$$\xrightarrow{2 + 2(3)} \left[\begin{array}{ccc|c} 1 & 0 & 7 & -1 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & -1 & 8 \end{array}\right] \xrightarrow{1 + 7(3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 55 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & -1 & 8 \end{array}\right]$$

$$\xrightarrow{-1(3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 55 \\ 0 & 1 & 0 & 15 \\ 0 & 0 & 1 & -8 \end{array}\right] \rightsquigarrow \begin{array}{l} x = 55 \\ y = 15 \\ z = -8 \end{array}$$

We can't always get matrices of this form, which we will see next week.

Practice problems: 1.1.2, 1.1.4, 1.1.9 (choose 2), 1.1.10 (choose 1), 1.1.15, 1.1.20